

A critical look at V_{us} determinations from hadronic τ decay data

Kim Maltman, Math and Statistics, York University, 4700 Keele St., Toronto, ON CANADA M3J 1P3*

A critical review of hadronic τ decay data based determinations of $|V_{us}|$ is given, focussing on the impact of the slow convergence of the integrated $D = 2$ OPE series for the conventional flavor-breaking sum rule determination and the potential role of as-yet-unmeasured multiparticle contributions to the strange spectral distribution. Additional information obtainable from analyses of inclusive strange decay data alone, and from mixed electroproduction- τ sum rules with much reduced OPE uncertainties, is also discussed. Self-consistency tests are shown to favor determinations which reduce somewhat discrepancies with 3-family unitarity expectations.

1. Introduction

$K_{\ell 3}$ and $\Gamma[K \rightarrow \mu\nu]/\Gamma[\pi \rightarrow \mu\nu]$ based determinations of $|V_{us}|$ [1] are in excellent agreement with expectations based on $|V_{ud}|$ [2] and 3-family unitarity. In contrast, determinations from flavor breaking (FB) sum rules involving hadronic τ decay data [3,4,5,6] yield results $\sim 3\sigma$ low, even after accounting for recent HFAG strange exclusive branching fraction (BF) updates [6].

For a kinematic-singularity-free correlator, Π , with spectral function, $\rho(s)$, the finite energy sum rule (FESR) relation on which the FB τ determination is based takes the form

$$\int_0^{s_0} w(s)\rho(s) ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s)\Pi(s) ds \quad (1)$$

a result valid for any s_0 and any analytic $w(s)$.

$|V_{us}|$ is obtained by applying Eq. (1) to the FB difference $\Delta\Pi_\tau \equiv [\Pi_{V+A;ud}^{(0+1)} - \Pi_{V+A;us}^{(0+1)}]$, where $\Pi_{V+A;ij}^{(J)}(s)$ are the spin $J = 0, 1$ components of the flavor ij , vector (V) or axial vector (A) current two-point functions, and $(0+1)$ denotes the sum of $J = 0$ and 1 components. For sufficiently large s_0 , the RHS of Eq. (1) is evaluated using the OPE representation, $[\Delta\Pi_\tau]^{OPE}$, while, for $s_0 \leq m_\tau^2$, the LHS is obtainable from inclusive hadronic τ decay distributions. Explicitly, the spectral

functions, $\rho_{V/A;ij}^{(J)}$, are related to the differential distributions, $dR_{V/A;ij}/ds$, of the normalized flavor ij V or A current induced decay widths, $R_{V/A;ij} \equiv \Gamma[\tau^- \rightarrow \nu_\tau \text{ hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e(\gamma)]$, by [7]

$$\frac{dR_{V/A;ij}}{ds} = c_\tau^{EW} |V_{ij}|^2 \left[w_\tau(y_\tau) \rho_{V/A;ij}^{(0+1)}(s) - w_L(y_\tau) \rho_{V/A;ij}^{(0)}(s) \right] \quad (2)$$

with $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, $w_L(y) = 2y(1-y)^2$, V_{ij} the flavor ij CKM matrix element, and, with S_{EW} a short-distance electroweak correction [8], $c_\tau^{EW} \equiv 12\pi^2 S_{EW}/m_\tau^2$.

The choice of FESRs involving the $J = 0 + 1$ combination, $\Delta\Pi_\tau$, rather than that corresponding to the linear combination of spectral functions appearing in Eq. (2), is predicated on the extremely bad behavior of the integrated $J = 0$, $D = 2$ OPE series for scales kinematically accessible in τ decay [9]. Fortunately, the dominant $J = 0$ spectral contributions are the accurately known, non-chirally-suppressed π and K pole terms. The remaining, continuum contributions, are (i) doubly chirally suppressed ($\rho_{V+A;ij}^{(0)} \propto [(m_i \mp m_j)^2]$), (ii) numerically negligible for $ij = ud$, and (iii) determinable phenomenologically for $ij = us$ $J = 0$ via dispersive [10] and sum rule [11] analyses of the strange scalar and pseudoscalar channels. The $J = 0$ contributions can thus be subtracted bin by bin from $dR_{V+A;ij}/ds$, allowing one to determine $\rho_{V+A;ij}^{(0+1)}(s)$ and construct the re-weighted $J = 0 + 1$ spectral integrals, $R_{V+A;ij}^w(s_0)$, and FB

*Alternate address: CSSM, Univ. of Adelaide, Adelaide, SA 5005 Australia; work supported by the Natural Sciences and Engineering Research Council of Canada

differences, $\delta R_{V+A}^w(s_0)$, defined by

$$\begin{aligned} \frac{R_{V+A;ij}^w(s_0)}{c_\tau^{EW}|V_{ij}|^2} &\equiv \int_0^{s_0} ds w(s) \rho_{V+A;ij}^{(0+1)}(s) , \\ \delta R_{V+A}^w(s_0) &= \frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2} - \frac{R_{V+A;us}^w(s_0)}{|V_{us}|^2} \\ &= c_\tau^{EW} \int_0^{s_0} ds w(s) \Delta \rho_\tau(s) . \end{aligned} \quad (3)$$

Taking $|V_{ud}|$ and any parameters in the OPE representation of δR_{V+A}^w from other sources, Eq. (1) then yields [3]

$$|V_{us}| = \sqrt{\frac{R_{V+A;us}^w(s_0)}{\frac{R_{V+A;ud}^w(s_0)}{|V_{ud}|^2} - \delta R_{V+A}^{w,OPE}(s_0)}} . \quad (4)$$

For weights used previously in the literature [3, 4,5], the OPE contribution to the denominator is at the few-to-several-% level of the ud spectral integral term, making modest accuracy for $\delta R_{V+A}^{w,OPE}(s_0)$ sufficient for a high accuracy determination of $|V_{us}|$. The smallness of the FB OPE corrections is illustrated by the results of the $s_0 = m_\tau^2$, w_τ FESR where, with updated HFAG τ BF's [6], supplemented by Standard Model (SM) $K_{\mu 2}$ expectations for the $\tau \rightarrow K \nu_\tau$ BF, B_K , the value $|V_{us}| = 0.2108(19)$ obtained ignoring FB OPE corrections differs from that obtained using one of the possible evaluations of $\delta R_{V+A}^{w_\tau,OPE}(m_\tau^2)$, $|V_{us}| = 0.2174(22)$ [6], by only $\sim 3\%$.

The task of quantifying the uncertainty in the OPE estimate for $\delta R_{V+A}^w(s_0)$ and, from this, the theoretical error on $|V_{us}|$, is complicated by the rather slow convergence, at the correlator level, of the series for the leading dimension $D = 2$ OPE contribution $[\Delta \Pi_\tau]_{D=2}^{OPE}$. To four loops [12]

$$\begin{aligned} [\Delta \Pi_\tau(Q^2)]_{D=2}^{OPE} &= \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[1 + \frac{7}{3} \bar{a} \right. \\ &\quad \left. + 19.93 \bar{a}^2 + 208.75 \bar{a}^3 + d_4 \bar{a}^4 + \dots \right] \end{aligned} \quad (5)$$

with $\bar{a} = \alpha_s(Q^2)/\pi$, and $\alpha_s(Q^2)$ and $m_s(Q^2)$ the running coupling and strange quark mass in the \overline{MS} scheme². Since $\bar{a}(m_\tau^2) \simeq 0.1$, convergence at the spacelike point on the contour $|s| = s_0$

²In what follows, we employ the estimate $d_4 \sim 2378$ [12] for the as-yet-undetermined 5-loop coefficient d_4 .

is marginal at best. With such slow convergence, conventional error estimates may significantly underestimate the $D = 2$ truncation uncertainty.

Fortunately, the FESR framework allows for internal self-consistency checks. Assuming both the data and OPE error estimates are reliable, the $|V_{us}|$ obtained from Eq. (4) should be independent of s_0 and $w(s)$. On the OPE side, three common methods can be employed for evaluating the integrated $D = 2$ contribution: the contour improved (CIPT) prescription, used with either (i) the truncated expression for $[\Delta \Pi_\tau]_{D=2}^{OPE}$ itself, or, (ii) after partial integration, with the correspondingly truncated Adler function, and (iii) the truncated fixed-order (FOPT) prescription (for which the correlator and Adler function results are identical). While the CIPT prescription corresponds to the conventional procedure of choosing a local scale, arguments in favor of the FOPT alternative also exist [13]. At a given truncation order, the three prescriptions differ only by contributions of yet higher order. If the estimated truncation errors are reliable, the results for $|V_{us}|$ obtained using the different prescriptions should thus also agree within these errors.

2. $|V_{us}|$ from FB hadronic τ decay FESRs

The results in this section are obtained using the updated December 2010 HFAG τ BF's [6], supplemented by SM $K_{\mu 2}$ expectations for B_K . SM $\pi_{\mu 2}$ expectations are similarly employed for the non-strange π pole contributions. Assuming lepton universality, the constrained HFAG determination of the electron BF, $B_e = 0.17852(27)$, implies $R_{V+A;ud} = 3.467(9)$ and $R_{V+A;us} = 0.1623(28)$. The change in $R_{V+A;ud}$ necessitates a small rescaling of the publicly available 2005 ALEPH ud distributions [14], from which the continuum contributions to $\rho_{V+A;ud}(s)$ are obtained. For the continuum part of $\rho_{V+A;us}$, the BaBar and Belle analyses of the inclusive strange distribution are not yet complete. The completed 1999 ALEPH distribution [15] corresponds to exclusive strange BF's with significantly larger errors, and, in many cases, significantly different central values, than those obtained by the B factory experiments [6]. As an interim measure, we fol-

low the strategy of Ref. [16], “partially updating” $\rho_{V+A;us}(s)$ to reflect the new values of the BFs through a mode-by-mode rescaling of the 1999 ALEPH distribution. This procedure, though not ideal, has been tested using the BaBar distribution data for $\tau \rightarrow K^- \pi^+ \pi^- \nu_\tau$ [17] (a mode having a particularly large BF change), and found to yield spectral integral contributions in surprisingly good agreement with those of the actual BaBar data. The rescaling method, however, has not been checked for other modes, and is likely to be less reliable for the $K3\pi$, $K4\pi$, \dots contributions, where the ALEPH distribution was not measured, but estimated using Monte Carlo, and has a purely phase space shape.

OPE input is as in the last of Refs. [4], except for the update $\alpha_s^{n_f=3}(m_\tau^2) = 0.3181(57)$, which reflects the new world average, $\alpha_s^{n_f=5}(M_Z^2) = 0.1184(7)$ [18].

We begin with the conventional w_τ determination. For $s_0 = m_\tau^2$, the ud and us spectral integrals are determined by the corresponding inclusive BFs. Improvements to the various exclusive strange BFs [6] then translate into an improved determination of the us spectral integral, even without completion of the remeasurement of the inclusive strange distribution. This is *not* the case for other s_0 and/or other $w(s)$. The $D = 2$ truncation uncertainty in this case has been estimated using standard last-term-retained \oplus residual-scale-dependence methods (known to work well for perturbative series displaying good convergence behavior). The resulting combined theoretical uncertainty on $|V_{us}|$ is then 0.0005 [5]. To test for the independence of $|V_{us}|$ on s_0 and $w(s)$, one must go beyond w_τ and/or $s_0 = m_\tau^2$. We perform these tests using the interim updated $\rho_{V+A;us}(s)$.

In Fig. 1, results for $|V_{us}|$ as a function of s_0 are shown for each of the three prescriptions for the w_τ -weighted $D = 2$ OPE series. The two CIPT-based prescriptions give similar results, but show poor s_0 -stability. The FOPT prescription yields significantly improved, though not perfect, s_0 -stability. For all s_0 the FOPT and CIPT results differ by significantly more than the 0.0005 total theoretical error estimate mentioned above.

The observed s_0 -instability of the FB w_τ FESR

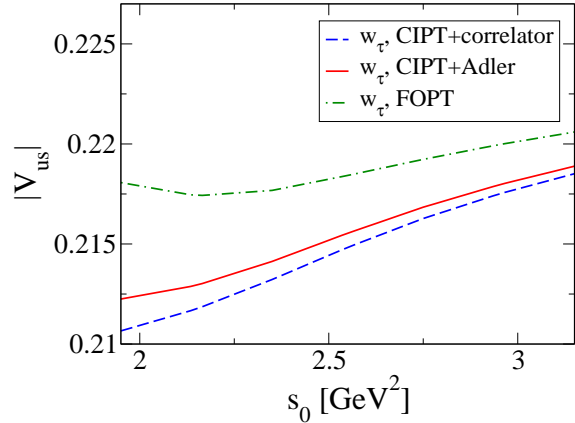


Figure 1. $|V_{us}|$ vs. s_0 from the FB w_τ -weighted FESR for the three different prescriptions for handling the integrated $D = 2$ OPE series.

results could arise from deficiencies in any of the contributions appearing in Eq. (4). Problems with the ud spectral integrals seem unlikely, but can be tested for using FESRs for the us V+A channel alone. A problem with missing higher multiplicity us spectral strength could also be exposed by such an analysis, assuming the OPE representation used to be reliable. Specifically, for s_0 large enough to include some portion of the region of missing strength, $|V_{us}|$ would be too low, while for s_0 below the region of the missing strength, $|V_{us}|$ should rise back to its true value. The us V+A FESRs are considered in the next section.

The most obvious candidate for a source of s_0 -instability in the OPE contribution is the potentially slowly converging integrated $D = 2$ series. For $s_0 = m_\tau^2$, the w_τ -weighted integrated $D = 2$ series behaves as $1 + 0.29 + 0.10 - 0.04 - (0.20) + \dots$ for the CIPT+Adler function case, $1 + 0.15 + 0.02 - 0.12 - (0.29) + \dots$ for the CIPT+correlator case, and $1 + 0.40 + 0.26 + 0.15 + (0.08) + \dots$ for the FOPT case, where the terms in parentheses are the estimated 5-loop, $O(\alpha_s^4)$ contributions. For the two CIPT cases, though there is

cancellation on the contour at intermediate orders, this cancellation does not persist to higher orders, and the integrated series appears rather badly behaved. The behavior is better for the FOPT case, which was also found to yield improved s_0 -stability. Worth noting is the fact that $\delta R_{V+A}^{w_\tau, OPE}(m_\tau^2)$ for the better behaved FOPT prescription is a factor of ~ 2 larger than that for either of the two CIPT prescriptions. The $\sim 100\%$ increase between CIPT and FOPT increases $|V_{us}|$ by ~ 0.0020 establishing the insufficiently conservative nature of the 0.0005 estimate for the combined theoretical error. With the difference between the FOPT result and the average of the two CIPT results as a new truncation uncertainty estimate, this source dominates the theoretical error. Taking the FOPT result (favored by the s_0 -stability criterion) as the new central value, the result for the FB w_τ determination becomes

$$|V_{us}| = 0.2193(3)_{ud}(19)_{us}(19)_{th}, \quad (6)$$

$\sim 2.3\sigma$ below 3-family unitarity expectations.

FB τ -based FESRs can also be constructed using other $w(s)$. For the CIPT prescription, if the s_0 -instability of the w_τ FESR results from premature truncation of the $D = 2$ series, convergence can be improved by choosing weights which emphasize contributions from the region of the complex $s = -Q^2$ plane away from the spacelike point, where $|\alpha_s(Q^2)|$ is smaller and the convergence, at the correlator level, of the $D = 2$ series better. Three weights (denoted $w_{10}(y)$, $\hat{w}_{10}(y)$, and $w_{20}(y)$, with $y = s/s_0$) having this, as well as other desirable properties, were constructed in Ref. [19]. The results for $|V_{us}|$ from the resulting FB FESRs, using the CIPT+correlator prescription, are shown in Fig. 2. The w_τ results for both the FOPT and CIPT+correlator prescriptions are shown for comparison. Improved s_0 -stability compared to the w_τ CIPT results is observed³. The $|V_{us}|$ values are compatible with

³FOPT-based results for the new weights are not shown, though they also display significantly improved s_0 -stability. The reason is that, having been constructed to improve the convergence of the integrated CIPT series, there is no reason to expect the new weights to also improve the convergence of the integrated FOPT series, and indeed they do not. For example, the FOPT version of the

the result of Eq. (6) within the theoretical errors quoted previously for that case. The central result for \hat{w}_{10} (the weight showing the best s_0 -stability) is $|V_{us}| = 0.2188$. The experimental error, which, absent a re-measurement of the inclusive strange distribution, has to be based on the 1999 ALEPH us covariances, is 0.0033.

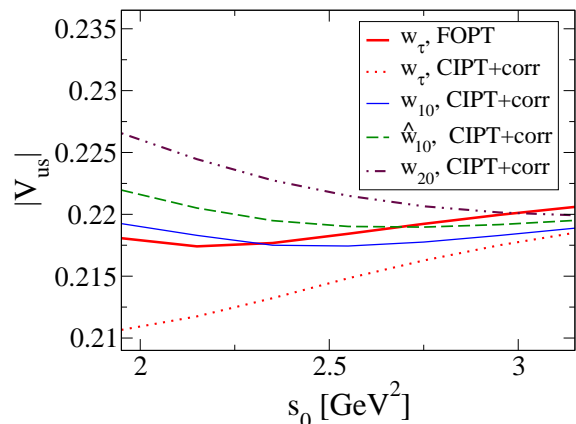


Figure 2. $|V_{us}|$ vs. s_0 , obtained using the CIPT+correlator $D = 2$ OPE prescription for the FB w_{10} , \hat{w}_{10} and w_{20} FESRs, and, for comparison, using FOPT and CIPT+correlator treatments for the w_τ FESR.

3. The us V+A and mixed τ -EM FESRs

FESR analyses of the us , $J = 0 + 1$, V+A correlator combination provide an alternate determination of $|V_{us}|$, one that provides some cross-checks on the FB FESR results. Two new ingredients enter the OPE side of these FESRs: $D = 0$ contributions, determined by α_s , and a $D = 4$ gluon condensate contribution absent from the

\hat{w}_{10} -weighted $D = 2$ series behaves as $1 + 0.51 + 0.43 + 0.39 + (0.41) + \dots$, in contrast to the CIPT+correlator version, which behaves as $1 + 0.24 + 0.19 + 0.15 + (0.11) + \dots$.

FB difference $\Delta\Pi_\tau$. It is known that, with the 5-loop $D = 0$ expansion as input [20], pinched FESR analyses of the ud , $J = 0 + 1$ V, A and V+A correlator combinations [21] yield values of α_s in excellent agreement with current high-precision lattice determinations [22] and the new world average [18]. A strong anti-correlation between α_s and the gluon condensate [21] implies $\langle\alpha_s G^2/\pi\rangle = 0.012 \text{ GeV}^4$ for the central α_s input noted above. Results for $|V_{us}|$ as a function of s_0 obtained from the w_τ -weighted FESR using each of three prescriptions for the integrated $D = 2$ series are presented in Fig. 3. The compatibility of the results for each prescription with those of the corresponding FB w_τ FESR is excellent. However, the s_0 -dependence of $|V_{us}|$ for the two CIPT prescriptions is clearly not compatible with the assumption that the $D = 2$ OPE representation is reliable and the instability in the FB w_τ FESR results for $|V_{us}|$ due solely to missing higher multiplicity us spectral strength. As for the FB FESR case, the FOPT $D = 2$ treatment produces improved, though not perfect, s_0 -stability.

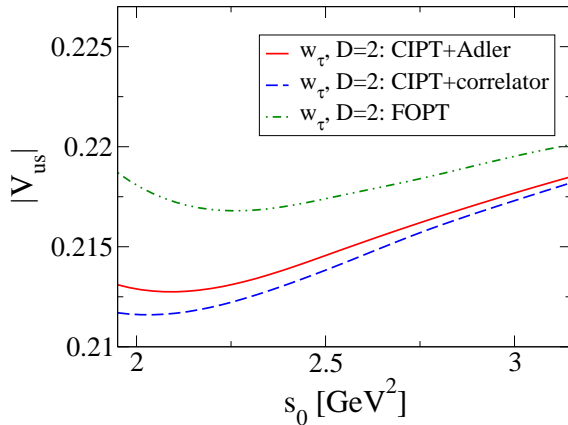


Figure 3. $|V_{us}|$ vs. s_0 for the w_τ -based us V+A FESR, using the three different prescriptions for handling the integrated $D = 2$ OPE series.

With problems in the FB τ FESRs, to at least some extent, due to slow $D = 2$ OPE convergence, FESRs with $D = 2$ OPE contributions reduced at the correlator level are highly desirable. With this in mind, combinations of $\Pi_{V/A;ud}^{(0+1)}$, $\Pi_{V+A;us}^{(0+1)}$ and the EM correlator, Π_{EM} , (whose spectral function, ρ_{EM} , is determined by the bare $e^+e^- \rightarrow \text{hadrons}$ cross-sections) can be constructed having zero $D = 0$ and vanishing $O(\alpha_s^0)$ $D = 2$ OPE contributions [23]. The unique such FB combination with the same normalization for $\Pi_{V+A;us}^{(0+1)}$ as $\Delta\Pi_\tau$ is

$$\Delta\Pi_M = 9\Pi_{EM} - 6\Pi_{V;ud}^{(0+1)} + \Delta\Pi_\tau \quad (7)$$

The $D = 2$ OPE series for $\Delta\Pi_M(Q^2)$ is

$$\frac{3}{2\pi^2} \frac{\bar{m}_s}{Q^2} \left[\frac{1}{3}\bar{a} + 4.38\bar{a}^2 + 44.9\bar{a}^3 + \dots \right] \quad (8)$$

which has not only (by construction) a vanishing $O(\alpha_s^0)$ coefficient, but also higher order coefficients significantly smaller than those for $\Delta\Pi_\tau$, *c.f.* Eq. (5). The resulting strong suppression of the $D = 2$ series is accompanied by a fortuitous suppression of $D = 4$ contributions. Explicitly, with $\delta_4 \equiv \langle m_s \bar{s}s - m_\ell \bar{\ell}\ell \rangle$, the $D = 4$ series for $\Delta\Pi_\tau$ and $\Delta\Pi_M$ have the form $\frac{\delta_4}{Q^4} \sum_k c_k \bar{a}^k$ with $(c_0, c_1, c_2) = (-2, -2, -26/3)$ for $\Delta\Pi_\tau$ and $(0, 8/3, 59/3)$ for $\Delta\Pi_M$ [24]. Integrated $D > 4$ contributions, which are expected to be somewhat enhanced [23], can be fitted to data using their stronger s_0 dependences.

A complication for the $\Delta\Pi_M$ FESRs is the fact that EM and τ results for the 2π and 4π components of the $I = 1$ V spectral function are not compatible at the level of expected isospin-breaking corrections [25]. We nonetheless present, as an illustration, the results for $|V_{us}|$ as a function of s_0 for the w_τ , $w_2(y) = (1-y)^2$ and $w_3(y) = 1 - \frac{3}{2}y + \frac{1}{2}y^3$ $\Delta\Pi_M$ FESRs, obtained assuming the τ data to be correct in both cases. Because of the current experimental complications, no attempt has been made to fit $D > 4$ OPE contributions. Instead, $D = 6$ contributions have been estimated using the vacuum saturation approximation and $D = 8$ contributions neglected. The main point is to illustrate the significantly

improved s_0 -stability produced by the strong suppression of the integrated $D = 2, 4$ OPE contributions. The $s_0 = m_\tau^2$, w_τ result in this case is

$$|V_{us}| = 0.2222(20)_\tau(28)_{EM} \quad (9)$$

where only the experimental errors have been displayed. While the very small theoretical errors for the mixed τ -EM FESRs make them promising for the future, further progress requires a resolution of the experimental 2π and 4π problems.

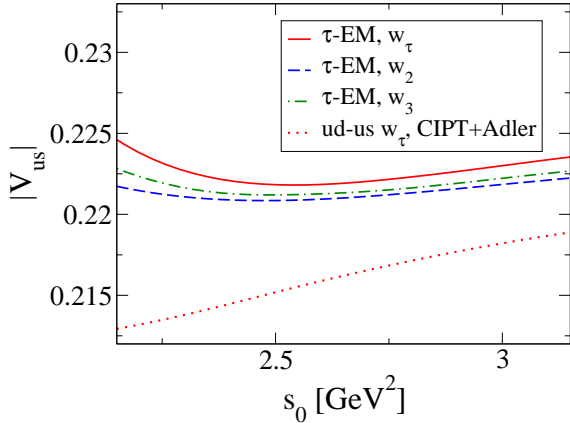


Figure 4. $|V_{us}|$ vs. s_0 for a selection of $\Delta\Pi_M$ FESRs and, for comparison, the CIPT+Adler w_τ $\Delta\Pi_\tau$ determination.

4. APPENDIX: Numerical results for the OPE input to the FB $s_0 = m_\tau^2$, w_τ FESR

This appendix contains numerical results for the OPE contributions to the conventional inclusive, non-longitudinally-subtracted, FB, $s_0 = m_\tau^2$ FESR determination of $|V_{us}|$, in a form that allows the reader to perform further explorations. For this case, in the notation of Ref. [5], one has

$$|V_{us}| = \sqrt{\frac{R_{\tau,S}}{\frac{R_{\tau,V+A}}{|V_{ud}|^2} - \delta R_{\tau,th}}}, \quad (10)$$

where $R_{\tau,S} \equiv R_{V+A;us}$ and $R_{\tau,V+A} \equiv R_{V+A;ud}$ are the B_e -normalized inclusive strange and non-strange branching fractions, defined above Eq. (2). The theoretical contribution, $\delta R_{\tau,th}$, is a sum of $J = 0$ (longitudinal) and $J = 0+1$ ($L+T$) contributions, denoted $\delta R_\tau|^L$ and $\delta R_{\tau,th}|^{L+T}$, respectively, in Ref. [5]. The former, which must be determined phenomenologically because of the problematic behavior of the longitudinal OPE representation, is dominated by the accurately known K and π pole contributions. Adding the current best assessment of the small continuum contributions yields the result

$$\delta R_\tau|^L = 0.1544 \pm 0.0037 \quad (11)$$

quoted in Ref. [5].

$\delta R_{\tau,th}|^{L+T}$ is evaluated using the OPE. The OPE representation is a sum of contributions, $\delta R_{OPE;D}^{L+T}$, of dimensions $D = 2, 4, 6, \dots$. $D > 6$ contributions are typically neglected in the literature. $\delta R_{OPE;D=2}^{L+T}$ is proportional to $m_s^2(2 \text{ GeV})$ and, for $S_{EW} = 1.0201$ [8], $m_\tau = 1.77677 \pm 0.00015 \text{ GeV}$ [6], $m_s(2 \text{ GeV}) = 94 \text{ MeV}$, and the current $n_f = 5$ world average $\alpha_s(M_Z^2) = 0.1184 \pm 0.007$ [18] (equivalent to the $n_f = 3$ result $\alpha_s(m_\tau^2) = 0.3181 \pm 0.0057$) has the values given in Table 1, as a function of the truncation order, n_T , and $D = 2$ scheme choice.

Table 1

$\delta R_{OPE;D=2}^{L+T}$ for various n_T and $D = 2$ evaluation schemes (CI+Ad, CI+co and FOPT denote the CIPT+Adler function, CIPT+correlator and FOPT schemes, respectively)

Scheme	n_T	$\delta R_{OPE;D=2}^{L+T}$
CI+Ad	3	0.0578 ∓ 0.0013
	4	0.0494 ∓ 0.0025
CI+co	3	0.0515 ∓ 0.0022
	4	0.0371 ∓ 0.0040
FOPT	3	0.1002 ± 0.0016
	4	0.1046 ± 0.0019

With ChPT input for $m_s/(m_d + m_u)$, GMOR for $\langle(m_d + m_u)\bar{u}u\rangle$, and the input already specified

above, the $D = 4$ contribution is given by

$$\delta R_{OPE;D=4}^{L+T} = 0.0072(2)r_c - 0.00030 \\ + 0.00060(11) \left(\frac{m_s(2 \text{ GeV})}{94 \text{ MeV}} \right)^4 \quad (12)$$

with $r_c \equiv \langle \bar{s}s \rangle / \langle \bar{u}u \rangle$, and the three terms on the RHS being associated with the strange quark condensate, light quark condensate and $O(m_s^4)$ contributions, respectively. The errors shown are those associated with the uncertainty in the input α_s . The value $r_c = 0.8$ employed in earlier analyses, follows from a 2002 sum rule analysis of f_{B_s}/f_B with then-current quenched lattice results for the decay constant ratio as input [27]. Current $n_f = 2 + 1$ results for this ratio yield instead $r_c = 1.2$. Finally, if desired, an estimate of the small $D = 6$ contribution can be included using a rescaled version of the vacuum saturation approximation (VSA). This yields

$$\delta R_{OPE;D=6}^{L+T} = -0.0043(1-r_c^2) \left(\frac{\rho \alpha_s \langle \bar{u}u \rangle^2}{0.000150 \text{ GeV}^6} \right) \quad (13)$$

where ρ parametrizes VSA breaking. The value in the denominator of the last factor reflects Ioffe's assessment of ρ [26].

REFERENCES

1. For a recent assessment and references to earlier work, see M. Antonelli *et al.*, Eur. Phys. J. C69 (2010) 399.
2. I.S. Towner and J.C. Hardy, Rep. Prog. Phys. 73 (2010) 046301.
3. E. Gamiz *et al.*, JHEP 0301 (2003) 060; Phys. Rev. Lett. 94 (2005) 011803.
4. K. Maltman and C.E. Wolfe, Phys. Lett. B639 (2006) 283; *ibid.* B650 (2007) 27; K. Maltman *et al.* Int. J. Mod. Phys. A23 (2008) 3191.
5. E. Gamiz *et al.*, PoS KAON 2007 (2008) 008; A. Pich, Nucl. Phys. Proc. Suppl. 181-182 (2008) 300.
6. D. Asner *et al.* (The HFAG Group) arXiv:1010.1589 [hep-ex].
7. Y.-S. Tsai, Phys. Rev. D4 (1971) 2821.
8. J. Erler, Rev. Mex. Fis. 50 (2004) 200.
9. K. Maltman, Phys. Rev. D58 (1998) 093015; K. Maltman and J. Kambor, *ibid.* D64 (2001) 093014.
10. M. Jamin, J.A. Oller and A. Pich, Nucl. Phys. B587 (2000) 331; *ibid.* B622 (2002) 279; and Phys. Rev. D74 (2006) 074009.
11. K. Maltman and J. Kambor, Phys. Rev. D65 (2002) 074013.
12. K.G. Chetyrkin and A. Kwiatkowski, Z. Phys. C59 (1993) 525 and hep-ph/9805232; P.A. Baikov, K.G. Chetyrkin and J.H. Kuhn, Phys. Rev. Lett. 95 (2005) 012003.
13. M. Beneke, M. Jamin, JHEP 0809 (2008) 044.
14. S. Schael *et al.* (The ALEPH Collaboration), Phys. Rep. 421 (2005) 191; M. Davier, A. Hocker and Z.Q. Zhang, Rev. Mod. Phys. 78 (2006) 1043.
15. R. Barate *et al.* (The ALEPH Collaboration), Eur. Phys. J. C11 (1999) 599.
16. S. Chen *et al.*, Eur. Phys. J. C22 (2001) 31.
17. B. Aubert *et al.* (The BaBar Collaboration), Phys. Rev. Lett. 100 (2008) 011801; I.M. Nugent, U. Victoria PhD thesis, SLAC-R-936.
18. S. Bethke, Eur. Phys. J. C64 (2009) 689.
19. J. Kambor and K. Maltman, Phys. Rev. D62 (2000) 093023.
20. P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 101 (2008) 012002.
21. See K. Maltman, T. Yavin, Phys. Rev. D78 (2008) 094020 and earlier references therein.
22. C.T.H. Davies *et al.*, Phys. Rev. D78 (2008) 114507; K. Maltman *et al.*, *ibid.* D78 (2008) 114504; E. Shintani *et al.*, *ibid.* D82 (2010) 074505; C. McNeile *et al.*, *ibid.* D82 (2010) 034512.
23. K. Maltman, Phys. Lett. B672 (2009) 257.
24. E. Braaten, S. Narison and A. Pich, Nucl. Phys. B373 (1992) 581.
25. See M. Davier *et al.*, arXiv:1010.4180 [hep-ph], and earlier references therein.
26. B.L. Ioffe and K.N. Zyablyuk, Eur. Phys. J. C27 (2003) 229; B.L. Ioffe, Phys. Atom. Nucl. 66 (2003) 30.
27. M. Jamin and B.O. Lange, Phys. Rev. D65: 056005 (2002); M. Jamin, Phys. Lett. B538, 71 (2002)